Jacobi's method (method of rotation)

Completely solves the problem for eigenvalues and eigenvectors of a symmetrical matrix. Practically used as a computer method. It uses the algebraic fact that if ***A*** **is symmetrical,** then there is an **orthogonal** matrix **U** ($U^T = U^{-1}$), for which $\boldsymbol{D = U^{-1}AU}$ **is a diagonal.** Also, if λ is the eigenvalue of $A \Rightarrow \lambda$ is the eigenvalue of D, then the eigenvalues λ match with the diagonal elements (d_{ii}) of D.

Algorithm

Let us assume that $A_0 = A$, $A_k = U_k^{-1}.A_{k-1}.U_k$, where U_k is a line of two-dimensional transformations ($U_1.U_2.U_3\ldots(= U)$), each of which has the form:

$$U_k = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & c & . & . & s & . & . & . \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & s & . & . & -c & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccccccccc} & & & m & & & l & & & \end{array} \end{array},$$

where $c^2 + s^2 = 1$; the non-diagonal elements, with the exception of the marked elements $\boldsymbol{c}$ and $\boldsymbol{s}$, are equal to 0; $\boldsymbol{c}$ and $\boldsymbol{s}$ can be treated as $\cos(\varphi)$ and $\sin(\varphi)$, where φ is the angle of rotation. If our goal is to make the element $\boldsymbol{a_{ml}}$ equal to 0, then $\boldsymbol{c}$ and $\boldsymbol{s}$ must be chosen using the formulas:

$$c = \sqrt{\frac{1+p}{2}}, \qquad s = \varepsilon\sqrt{\frac{1-p}{2}},$$

where $\quad p = \sqrt{1 - \dfrac{(2a_{ml})^2}{(a_{mm} - a_{ll})^2 + (2a_{ml})^2}}$ and $\varepsilon = \begin{cases} sign(a_{ml}), & a_{mm} = a_{ll} \\ sign\dfrac{a_{mm} - a_{ll}}{a_{ml}}, & a_{mm} \neq a_{ll} \end{cases}.$

Then we calculate the natrix $B = U_k^{-1}.A.U_k$ with changed ***m*** and ***l*** lines and columns:

$$b_{mm} = c^2 a_{mm} + s^2 a_{ll} + 2csa_{ml}, \quad b_{ll} = s^2 a_{mm} + c^2 a_{ll} - 2csa_{ml},$$

$$b_{ml} = b_{lm} = 0,$$

$$b_{mj} = b_{jm} = ca_{jm} + sa_{jl}, \quad b_{lj} = b_{jl} = sa_{jm} - sa_{jl}, \quad j = 1,2,...,n, \quad j \neq m, \quad j \neq l,$$

$b_{jm} = a_{jm}$ in the rest of the cases.

Note. The last formulas for the elements of B are to be used while programming the method. Jacobi's method demands a large number of calculations, but for a computer it is not obligatory to choose for a_{ml} the modularly largest non-diagonal element (that is for calculating manually), whereas these elements are consecutively processed. On the other hand the method is reliable and applicable even in cases of multiple roots.

Example. Calculate the eigenvalues and eigenvectors of the matrix with an accuracy of $\varepsilon = 10^{-1}$, where

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 3 & 5 \\ 5 & 5 & -1 \end{bmatrix}.$$

Solution:

We derive the following diagram (the calculations are with a few symbols after the decimal point):

m, l	c, s	U_k	A_k
1, 3	0,828 0,561	$\begin{pmatrix} 0,828 & 0 & 0,561 \\ 0 & 1 & 0 \\ 0,561 & 0 & -0,828 \end{pmatrix}$	$\begin{pmatrix} 6,387 & 3,633 & 0,004 \\ 3,633 & 3 & 3,579 \\ 0,004 & 3,579 & -4,386 \end{pmatrix}$
1, 2	0,843 0,538	$\begin{pmatrix} 0,843 & 0,538 & 0 \\ 0,538 & -0,843 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 8,703 & 0,006 & -1,922 \\ 0,006 & 0,685 & 3,019 \\ -1,922 & 3,019 & -4,386 \end{pmatrix}$

2, 3	0,906 0,422	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0{,}906 & 0{,}422 \\ 0 & 0{,}422 & -0{,}906 \end{pmatrix}$	$\begin{pmatrix} 8{,}703 & -0{,}806 & 1{,}744 \\ -0{,}806 & 2{,}090 & -0{,}002 \\ 1{,}744 & -0{,}002 & -5{,}787 \end{pmatrix}$
1, 3	0,993 0,118	$\begin{pmatrix} 0{,}993 & 0 & 0{,}118 \\ 0 & 1 & 0 \\ 0{,}118 & 0 & -0{,}993 \end{pmatrix}$	$\begin{pmatrix} 8{,}910 & -0{,}801 & 0{,}002 \\ -0{,}801 & 2{,}090 & -0{,}093 \\ 0{,}002 & -0{,}093 & -5{,}994 \end{pmatrix}$
1, 2	0,993 -0,114	$\begin{pmatrix} 0{,}993 & -0{,}114 & 0 \\ -0{,}114 & -0{,}993 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 8{,}994 & 0{,}007 & 0{,}013 \\ 0{,}007 & 1{,}995 & 0{,}092 \\ 0{,}013 & 0{,}092 & -5{,}994 \end{pmatrix}$

m ,l in this case are indices of max($|a_{i,j}|$) at $i \neq j$. The calculations are done until the non-diagonal elements become equal to 0 with an accuracy of ε. Along the diagonal are the eigenvalues with an accuracy of ε, therefore $\lambda_1 \approx 9{,}0$; $\lambda_2 \approx 2{,}0$; $\lambda_3 \approx -6{,}0$. The eigenvectors of A are approximately equal to the columns of the matrix $U_1.U_2.\ldots U_k$, corresponding to λ_i:

$$U_1 U_2 \ldots U_5 = \begin{bmatrix} 0{,}6 & -0{,}7 & -0{,}4 \\ 0{,}6 & 0{,}7 & -0{,}4 \\ 0{,}6 & 0 & 0{,}8 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} -$$

is the matrix of normalized vectors.

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